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LETTER TO THE EDITOR

On the q oscillator and the quantum algebra $su_q(1,1)$

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Abstract. We discuss some problems of the q-deformed harmonic oscillator. Different q bosonisations of the quantum $su_q(1,1)$ algebra are given and the corresponding infinite dimensional representations of discrete series are analysed.

The development of the quantum inverse problem method [1] and the study of solutions to the Yang–Baxter equation [2] gave rise to the notion of quantum groups and algebras (cf [3, 4] and references therein). The growing interest in the subject is connected with the fact that the properties of quantum groups and algebras are quite similar to those of Lie groups and Lie algebras in connection with both the representation theory and the possible physical applications. In particular, quantum superalgebras [5, 6] and the q deformation of the quantum harmonic oscillator [7–9] were defined. It was demonstrated that the q deformation of the oscillator (q oscillator) can be used in the quantum algebra representation theory in the same manner as in the case of the harmonic oscillator [7–9].

In this letter we discuss the theory of the q oscillator, constructing for the q annihilation operator a shift operator which generates q-coherent states. We formulate some important yet unsolved problems concerning the q oscillator which seem to be more complicated than those of the harmonic oscillator. One of them is the question of equivalence of the associative algebras $\mathscr{A}(q)$ generated by the q oscillators with different values of q. For the Fock representation we find the expression for the q oscillator in terms of the boson operators. Alternative forms of the q commutation relations are given and a relationship between different multimode generalisations [6, 10] is established. We propose a realisation of the quantum sl(2) generators in terms of one q oscillator instead of two as used in [7–9]. The corresponding infinite-dimensional representations are reducible. The irreducible components of their decompositions belong to the discrete series of the quantum algebra su(1, 1).

Let us consider an associative algebra $\mathscr{A}(q)$ with units generated by three elements a, a^+, N which satisfy the relations

$$[N, a^+] = a^+ \qquad [N, a] = -a \tag{1}$$

and

$$aa^+ - qa^+a = q^{-N}. \tag{2}$$

To introduce $*$ operation in $\mathscr{A}(q)$ we suppose $q \in \mathbb{R}$:

$$(a)^* = a^+ \qquad (a^+)^* = a \qquad N^* = N. \tag{3}$$

Then the relations (1) and (2) are invariant with respect to the $*$ anti-involution. Within the pure algebraic frame one can use instead of N the generators k^+, k^- with relations $[k^\pm = \exp(\pm N \ln q)]$:

$$k^\pm a = q^{\mp 1} a k^\pm \qquad k^+k^- = k^-k^+ = 1 \qquad (k^\pm)^* = k^\pm. \tag{4}$$

However we prefer the generator N with simple physical interpretation (the excitation number operator for the Fock space representation cf below).

There exist different forms of the relations (1) and (2). The modified operators [6] $\alpha = q^{-N/2}a$, $\alpha^+ = a^+ q^{-N/2}$ satisfy (1) and the commutation relation

$$[\alpha, \alpha^+] = q^{-2N} \tag{5}$$

The operators $A = q^N \alpha$, $A^+ = \alpha^+ q^N$ satisfy (1) and

$$AA^+ - \mu A^+ A = 1 \qquad (\mu = q^2). \tag{6}$$

As far as we know the latter form with $\mu \in [-1, \infty)$ was first introduced in [11] where the Fock space representation was described. Later on it was used in so-called hadronic mechanics and Lie-admissible algebras (cf [12] and references therein). Recently the q oscillator (in different forms) was rediscovered by different authors [7-10] in connection with quantum groups and algebras. One can construct the representation of the relations (1), (2) (or (5), (6)) in the Fock space $\mathscr{H}_F$ spanned on normalised eigenstates $|n\rangle$ of the excitation number operator N:

$$a|0\rangle = 0 \qquad N|n\rangle = n|n\rangle \qquad n = 0, 1, 2, \ldots \tag{7}$$

$$|n\rangle = ([n]_1!)^{-1/2}(a^+)^n|0\rangle \tag{8}$$

$$[n]_q! = [1], [2], \ldots, [n] \qquad [n]_q \equiv [n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{9}$$

One can express $|n\rangle$ in terms of α^+ or A^+ operators with slightly different coefficients. The basis (8) is orthonormal due to the formulae $(\mu = q^2)$:

$$a(a^+)^n = (qa^+)^n a + [n]_q (a^+)^{n-1} q^{-N} \tag{10}$$

$$\alpha(\alpha^+)^n = (\alpha^+)^n \alpha + [n]^c_{\mu^{-1}} (\alpha^+)^{n-1} \mu^{-N} \tag{11}$$

$$A(A^+)^n = (\mu A^+)^n A + [n]^c_\mu (A^+)^{n-1} \tag{12}$$

where we also use the notation from the q analysis [13]

$$[n]^c_\mu = (\mu^n - 1)/(\mu - 1) = q^{n-1}[n]_q \qquad \mu = q^2. \tag{13}$$

In the Fock space $\mathscr{H}_F$ it is possible to express the q oscillator in terms of the usual Bose operators b, b^+:

$$a = \left(\frac{[N+1]}{N+1}\right)^{1/2} b \qquad a^+ = b^+ \left(\frac{[N+1]}{N+1}\right)^{1/2}. \tag{14}$$

Similar expressions for the quantum $\mathrm{sl}_q(2)$ generators in terms of the Lie algebra sl(2) generators were constructed in [14].

As a result we obtain from (14), $[b, b^+] = 1$ and $N = b^+ b$ for a, a^+ the relations (1), (2) or the relation with q^{-1}:

$$aa^+ - q^{-1} a^+ a = q^N. \tag{15}$$

It follows from (2), (15) or (14) that in $\mathscr{H}_F$ there exist the equalities [7, 9]

$$a^+ a = [N] = \frac{q^N - q^{-N}}{q - q^{-1}} \qquad aa^+ = [N+1]. \tag{16}$$

Is it possible to obtain (16) for a, a^+ using the relations (1), (2) only? Algebraic manipulations show that the operator

$$C=[N]-a^+a \tag{17}$$

commutes with a in a different way in the cases of relations (1), (2) or (1), (15) respectively:

$$aC=qCa \qquad aC=q^{-1}Ca. \tag{18}$$

They are the same in $\mathscr{H}_F$ because $C=0$ due to (16).

Taking into account (14) it is easy to conclude that in $\mathscr{H}_F$ the basis $|n\rangle$ is nothing but the well known n-boson states. This is not so for the corresponding coherent states, which can be constructed for the operators a (α or A) [12, 8]

$$a|z\rangle=z|z\rangle \qquad |z\rangle=(\mathcal{N}(z))^{-1/2}e_q(za^+)|0\rangle \tag{19}$$

$$\mathcal{N}(z)=e_q(|z|^2) \qquad e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_q!}. \tag{20}$$

In these formulae the notation of q analysis for the q exponent is used. The same set of states (19) can be obtained introducing an operator T which satisfies the commutation relation

$$[a, T]=1 \tag{21}$$

and the displacement group $\mathscr{D}(z)=\exp zT$ for a:

$$a\mathscr{D}(z)=\mathscr{D}(z)(a+z) \qquad |z\rangle=(\mathcal{N}(z))^{-1/2}\mathscr{D}(z)|0\rangle. \tag{22}$$

These coherent states for $q\neq 1$ do not coincide with the usual states and we postpone the discussion of their completeness until the next publication.

There exists an analogue of the holomorphic representation for the q oscillator (also known in q analysis [13]) where a^+ is the multiplication operator of x and a is the q differentiation operator D:

$$(Df)(x)=\frac{f(qx)-f(q^{-1}x)}{(q-q^{-1})x} \qquad Nx^n=nx^n. \tag{23}$$

Let us consider multimode q oscillators. In [7-9, 6] n independent (mutually commuting) q oscillators a_i, a_j^+, $i, j=1, 2, \ldots, n$ were used. Another set of operators φ_i, φ_j^+ was introduced in [10]. This system satisfies the relations ($i<j$)

$$\varphi_i^+\varphi_j^+=q\varphi_j^+\varphi_i \qquad \varphi_j\varphi_i=q\varphi_i\varphi_j$$

$$\varphi_k\varphi_l^+=q\varphi_l^+\varphi_k \qquad l\neq k$$

$$\varphi_i\varphi_i^+-q^2\varphi_i^+\varphi_i=1+(q^2-1)\sum_{k>i}\varphi_k^+\varphi_k.$$

the operators φ_i, φ_j^+ can be expressed in terms of independent q oscillators ($\varphi_j^+=(\varphi_j)^+$):

$$\varphi_i=q^{\Sigma_{k>i}N_k}q^{N_i/2}a_i=q^{\Sigma_{k>i}N_k}A_i$$

$$1+(q^2-1)\sum_{k>i}\varphi_k^+\varphi_k=\prod_{k>i}(1+(q^2-1)A_k^+A_k).$$

Because the relations (1) and (2) depend on two parameters q and Planck's constant $\hbar$ ($a^\pm\to a^\pm/\hbar^{1/2}$, $N\to N\hbar^{-1}$, $q=\exp\gamma\hbar$) one can consider different limits: $q\to 1$ ($\gamma\to 0$)

reproducing the Heisenberg algebra and $\hbar \to 0$ (γ is fixed) obtaining the classical system with one degree of freedom and the Poisson bracket for complex coordinates

$$\{a^*, a\} = \mathrm{i}(1+\gamma^2(a^*a)^2)^{1/2}. \tag{24}$$

We would like to mention that a difficult problem is the determination of the invariance group of the q oscillator. The relations (1), (2) or (24) are not invariant with respect to the translations or the Bogoliubov transformations. However, the q oscillator can be obtained from the finite dimensional representations of the quantum sl(2) algebra ($\mu = q$):

$$[J_0, J_\pm] = \pm J_\pm, [J_+, J_-] = [2J_0]_\mu \tag{25}$$

by the contraction procedure [6, 15].

The realisation of the generators (25) in terms of two independent q oscillators was given in [7-9]:

$$J_0 = \tfrac{1}{2}(N_1 - N_2) \qquad J_+ = a_1^+ a_2, \qquad J_- = a_2^+ a_1. \tag{26}$$

The relations of the quantum SU(1, 1) algebra have opposite signs in the second formula of (25)

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -[2K_0]_\mu \tag{27}$$

with the Casimir operator

$$c = [K_0 - \tfrac{1}{2}]_\mu^2 - K_+K_-. \tag{28}$$

It is possible to realise these generators in the Fock space $\mathscr{H}_\mathrm{F}$ of one q oscillator ($\beta = (q+q^{-1})^{-1}$)

$$K_0 = \tfrac{1}{2}(N + \tfrac{1}{2}) \qquad K_+ = \beta(a^+)^2 \qquad K_- = \beta a^2 \qquad \mu = q^2. \tag{29}$$

The corresponding representation in $\mathscr{H}_\mathrm{F}$ is reducible because the operator $P = (-1)$ commutes with all the generators (29). The decomposition of $\mathscr{H}_\mathrm{F}$ has two irreducible components with eigenvalues $(-1)^\varepsilon$, $\varepsilon = 0, 1$ of the operator P:

$$\mathscr{H}_\mathrm{F} = \mathscr{H}_0 \oplus \mathscr{H}_1 \tag{30}$$

consisting of the vectors with even and odd numbers of q-oscillator excitations.

In the case of two q oscillators one can easily construct two realisations of the $\mathrm{su}_\mu(1, 1)$ algebra (27). One is similar to (26) while the other is connected with (29) and the existence of a comultiplication for the su_μ (1, 1) [3, 4] or the possibility to define the action of the $\mathrm{su}_\mu(1, 1)$ generators in the tensor product of representation spaces.

If we define K_0, $K_\pm$ as

$$K_0 = \tfrac{1}{2}(N_1 + N_2 + 1) \qquad K_+ = (K_-)^+ = a_1^+ a_2^+ \qquad \mu = q \tag{31}$$

then these generators satisfy (27). As a result we have a reducible representation of the quantum algebra $\mathrm{SU}_\mu(1, 1)$ in the space $\mathscr{H}_\mathrm{F}^{(1)} \otimes \mathscr{H}_\mathrm{F}^{(2)}$ because the operator $N_1 - N_2$ commutes with K_0, $K_\pm$ (31). The decomposition of this representation into irreducible ones is

$$\mathscr{H}_\mathrm{F}^{(1)} \otimes \mathscr{H}_\mathrm{F}^{(2)} = \sum_{l=-\infty}^{\infty} \mathscr{H}_l \tag{32}$$

where the basis of the subspace $\mathscr{H}_l$ consists of the vectors $\{|n_1\rangle \otimes |n_2\rangle$, $n_1 = n_2 = l$, $n_1, n_2 = 0, 1, 2, \ldots\}$.

If we use a coproduct Δ: $\mathrm{SU}_\mu(1, 1) \to \mathrm{SU}_\mu(1, 1) \otimes \mathrm{SU}_\mu(1, 1)$ [3, 4]:

$$\Delta(K_0) = K_0 \otimes I + I \otimes K_0 \qquad \Delta(K_\pm) = K_\pm \otimes \mu^{K_0} + \mu^{-K_0} \otimes K_\pm \tag{33}$$

together with the notation $a_1 = a \otimes I$, $a_2 = I \otimes a$ and (29) then we can define another representation of $SU_\mu(1,1)$ in the space (32)

$$\begin{aligned} K_0 &= \tfrac{1}{2}(N_1+N_2+1) \qquad \mu = q^2 \\ K_+ &= \beta((a_1^+)^2 q^{N_2+1/2} + q^{-N_1-1/2}(a_2^+)^2) \\ K_- &= \beta((a_1)^2 q^{N_2+1/2} + q^{-N_1-1/2}(a_2)^2). \end{aligned} \tag{34}$$

Now the decomposition of $\mathscr{H}_F \otimes \mathscr{H}_F$ into irreducible representations has only one multiplicity free component. This decomposition is

$$\mathscr{H}_F \otimes \mathscr{H}_F = \mathscr{H}_{1/2} \oplus \mathbb{C}^2 \otimes \sum_{s=1}^{\infty} (\mathscr{H}_s \oplus \mathscr{H}_{s+1/2}) \tag{35}$$

where we parametrise the irreducible subspaces $\mathscr{H}_s$ by the eigenvalue of K_0 on the corresponding lowest weight vector:

$$K_0|s,0\rangle = s|s,0\rangle \qquad K_-|s,0\rangle = 0$$

$$|s,n\rangle \simeq (K_+)^n|s,0\rangle \qquad n = 0,1,2,\ldots.$$

In the three cases (29), (31), (34) we have the q-oscillator realisations of the $su_\mu(1,1)$ irreducible representations which belong to the discrete series. Formal construction of other series can be found for example in [16].

As in the su(1, 1) it is possible to construct coherent states of the quantum algebra $su_\mu(1,1)$ with corresponding modifications (cf (19), (20)). The quantum algebras can be used for the description of the dynamical symmetries of physical systems. In particular, the operators of the q-oscillator type, such as (14), were used in the quantum discrete integrable model which is the analogue of the one-dimensional nonlinear Schrödinger equation [17]. Work along these lines is in progress.

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